

# The effects of anti-correlation on gravitational clustering.

Nissim Kanekar<sup>1</sup> <sup>★</sup> and T. Padmanabhan<sup>2</sup> <sup>†</sup>

<sup>1</sup>*National Centre for Radio Astrophysics,*

*Post Bag 3, Ganeshkhind, Pune 411 007, India*

<sup>2</sup>*Inter-University Centre for Astronomy and Astrophysics,*

*Post Bag 4, Ganeshkhind, Pune 411 007, India*

## ABSTRACT

We use non-linear scaling relations (NSRs) to investigate the effects arising from the existence of negative correlations on the evolution of gravitational clustering in an expanding universe. It turns out that such anti-correlated regions have important dynamical effects on *all* scales. In particular, the mere existence of negative values for the linear two-point correlation function  $\xi_L$  over some range of scales starting from  $l = L_o$ , implies that the non-linear correlation function is bounded from above at *all* scales  $x < L_o$ . This also results in the relation  $\bar{\xi} \propto x^{-3}$ , at these scales, at late times, independent of the original form of the correlation function. Current observations do not rule out the existence of negative  $\bar{\xi}$  for  $200h^{-1}$  Mpc  $\bar{\xi}1000h^{-1}$  Mpc; the present work may thus have relevance for the real Universe. The only assumption made in the analysis is the *existence* of NSR; the results are independent of the form of the NSR as well as of the stable clustering hypothesis.

**Key words:** Cosmology : theory – dark matter, large scale structure of the Universe

## 1 INTRODUCTION : THE IMPORTANCE OF BEING NEGATIVE

The problem of gravitational clustering in an expanding Universe is one of the important open issues in cosmology. In the standard picture, non-linear structures were formed through the mechanism of gravitational instability, via the amplification of small initial density perturbations. At early epochs, when the fluctuations are in the linear regime, perturbation theory can be used to study the evolution. Analytical understanding of the quasi- and non-linear regimes has unfortunately proved a more intractable problem and much of the activity in this area has centred on numerical simulations. It is, however, important to *understand* the numerical results; a number of attempts have hence been made to address the subject by semi-analytic means. While this program is still in its infancy and we are far from providing a “first principles” paradigm, such attempts have nevertheless

been successful in identifying at least some of the key issues in the study of gravitational clustering.

A key problem in this area involves understanding how power gets transferred between different scales during the evolution of clustering. In the simplest context, one would like to understand how an initial power spectrum, peaked around a particular scale, evolves with time. Numerical results (Bagla & Padmanabhan 1997) and a few analytic insights (especially regarding inverse cascade; see Padmanabhan 2000) show that power injected at a scale  $L$  cascades both downwards (to  $x < L$ ) and upwards (to  $x > L$ ). To the lowest order of approximation, the Fourier transform of the gravitational potential is described by the equation (Padmanabhan 2000):

$$\ddot{\phi}_{\mathbf{k}} + 4\frac{\dot{a}}{a}\dot{\phi}_{\mathbf{k}} = -\frac{1}{3a^2} \int \frac{d^3\mathbf{p}}{(2\pi)^3} \phi_{\frac{1}{2}\mathbf{k}+\mathbf{p}} \phi_{\frac{1}{2}\mathbf{k}-\mathbf{p}} \mathcal{G}(\mathbf{k}, \mathbf{p}) \quad (1)$$

$$\mathcal{G}(\mathbf{k}, \mathbf{p}) = \frac{7}{8}k^2 + \frac{3}{2}p^2 - 5 \left( \frac{\mathbf{k} \cdot \mathbf{p}}{k} \right)^2,$$

which governs the growth of  $\phi_{\mathbf{k}}$  due to non-linear mode coupling. When the non-linear coupling term on the right hand side is small, we get the standard result that  $\phi_{\mathbf{k}}$  is inde-

<sup>★</sup> nissim@ncra.tifr.res.in

<sup>†</sup> paddy@iucaa.ernet.in

pendent of time in the linear regime. If, however, the initial power spectrum is sharply peaked at some scale, the issue becomes more subtle. Linear theory remains valid at all scales only as long as  $\phi_{\mathbf{k}}$  is small at *all* scales. As soon as some scale goes non-linear, it is possible for the integral on the right hand side of equation (1) to pick up significant contributions to drive the evolution of  $\phi_{\mathbf{k}}$  at *other* scales. For example, this will lead to a  $k^4$  tail in the power spectrum of the density contrast (Padmanabhan 2000), purely due to non-linear coupling. This is a clear case of inverse cascade of power in gravitational clustering.

Power transfer also occurs to smaller scales but is somewhat more difficult to analyse from first principles. Simulations (Bagla & Padmanabhan 1997) as well as theoretical arguments (Padmanabhan & Engineer 1998) suggest that the evolution to smaller scales will lead to a universal form of the power spectrum, reminiscent of the Kolmogorov spectrum in fluid turbulence, if the initial spectrum was sharply peaked at some scale.

The above results emphasise the simplicity and tractability of a question involving the evolution of a power spectrum which was originally peaked around some scale. This issue is important in understanding gravitational clustering, in spite of the fact that currently fashionable models for structure formation have broad band power spectra, with  $P(k) \propto k$  for small  $k$ . To study the evolution of such power spectra analytically, we need to use some technique which is sufficiently general and yet tractable. One such approach is based on non-linear scaling relations (NSRs, hereafter), which provide a mapping between the linear and non-linear mean two-point correlation functions (Hamilton et al. 1991; Nityananda & Padmanabhan 1994; Mo et al. 1995; Padmanabhan 1996a). The above mapping appears to be validated by numerical studies, which have also shown it to be fairly model independent. Given an initial mean correlation function,  $\bar{\xi}_L$ , in the linear regime, the NSR yields the evolved mean correlation function,  $\bar{\xi}$ , (and thus, the power spectrum) at any later time. It also illustrates how power is transferred between different scales during the evolution of clustering. Finally, the limiting forms of the NSR can be derived using the standard paradigms of scale-invariant radial collapse and stable clustering in the quasi-linear and non-linear regimes respectively (Padmanabhan 1996a); it has also been shown that scaling laws are a generic feature of clustering in an arbitrary number of dimensions (Padmanabhan & Kanekar 2000).

Non-linear scaling relations have thus proved a powerful tool for understanding gravitational clustering; however, they have one major defect. The standard NSR (Hamilton et al. 1991), by its very form, is restricted in its applicability to positive semi-definite correlation functions; negative values of  $\bar{\xi}$  are not included in its gamut. However, any sufficiently localised power spectrum (for example, a Dirac delta function), will produce negative  $\bar{\xi}$  at some scales; in fact, as will be seen later, negative  $\bar{\xi}$  can also arise from broad band power spectra. It is thus interesting, from a purely

mathematical perspective, to investigate the effects of anti-correlation on the formation of structure.

The standard NSR shows that power is transferred from large scales to small scales during the evolution. It turns out, however, that the very existence of negative  $\bar{\xi}$ , combined with the equation of conservation of pairs, implies that the flow of power will *reverse* direction at the scale at which  $\bar{\xi}$  crosses zero. Further, it will be shown that the transfer of power to small scales saturates at late times; this results in an upper bound on  $\bar{\xi}$  at all scales smaller than the one at which  $\bar{\xi}$  is initially negative. Finally, the correlation function at these scales satisfies the relation  $\bar{\xi} \propto x^{-3}$ , at late times, independent of its original form.

None of the above features arise for initial power spectra with positive definite correlation functions. It should be stressed, however, that these results are valid even if  $\bar{\xi}$  is only negative over a very small range of scales or at arbitrarily large scales; the results also do not depend on the *extent* to which  $\bar{\xi}$  goes negative but arise entirely due to the *existence* of negative  $\bar{\xi}$ . They are also quite independent of the stable clustering ansatz. Current observations do not preclude negative values of the correlation function at some scales; in fact, we will show in section 4 that one can indeed modify the power spectrum of the universe, so as to have negative correlation functions over certain ranges of scales, *without violating any observational constraints*.

The plan of the paper is as follows : the mathematical framework is set up in section 2; we also discuss the usual NSR for the positive  $\bar{\xi}$  regime here. Next, NSRs for other statistical indicators of clustering, such as the power spectrum, are considered in section 3. We move on to review observational constraints on the correlation function and the power spectrum in section 4, demonstrating (by example) that it is possible to set up power spectra yielding negative correlation functions without violating any observational constraints. Section 5 considers the effects of negative correlation functions on the evolution of clustering and shows that the existence of anti-correlated regions yields an upper bound on the correlation function at scales shortward of the scale at which the linear  $\bar{\xi}$  goes negative. Finally, we use a synthetic NSR (whose form is derived in an Appendix) in section 6, to demonstrate the above results.

## 2 MATHEMATICAL PRELIMINARIES

We will consider the evolution of a system starting from Gaussian initial conditions, with an initial power spectrum  $P_{\text{in}}(k)$ . The two-point correlation function,  $\xi(a, \mathbf{x})$ , is defined as the Fourier transform of the power spectrum :

$$\xi(a, \mathbf{x}) = \int \frac{d^3\mathbf{k}}{(2\pi)^3} P(a, \mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}} , \quad (2)$$

where  $a$  is the expansion factor of the Universe ( $a \propto t^{2/3}$  for an  $\Omega = 1$  matter-dominated Universe). It is usually more

convenient to work with the correlation function averaged over a sphere of radius  $x$ , given by

$$\bar{\xi}(a, x) = \frac{3}{x^3} \int_0^x \xi(a, y) y^2 dy. \quad (3)$$

The power spectrum and the mean correlation function are related by the equations:

$$\bar{\xi}(a, x) = \frac{3}{2\pi^2 x^3} \int_0^\infty \frac{dk}{k} P(a, k) [\sin(kx) - kx \cos(kx)] \quad (4)$$

and

$$P(a, k) = \frac{4\pi}{3k} \int_0^\infty dx x \bar{\xi}(a, x) [\sin(kx) - kx \cos(kx)]. \quad (5)$$

In the linear regime, it can be shown that  $\bar{\xi} \propto a^2$  so that  $\bar{\xi}_L(a, x) \propto a^2 \bar{\xi}_{\text{in}}(a_i, x)$ .

Note that the average two-point correlation function for a power spectrum given by  $P(k) = \delta_D(k - k_o)$ , where  $\delta_D$  represents the Dirac delta function is

$$\bar{\xi}(l) = \frac{3}{2\pi^2 l^3 k_o} [\sin(k_o l) - k_o l \cos(k_o l)], \quad (6)$$

which is negative over several ranges of scales, crossing zero for the first time when  $\tan(k_o l) = k_o l$ , *i.e.*  $l \sim 4.493/k_o$ . In fact, negative correlation functions always arise for sufficiently peaked functional forms of the power spectrum.

The two-point correlation function satisfies the equation of conservation of pairs (Peebles 1980):

$$\frac{\partial \xi}{\partial t} + \frac{1}{ax^2} \frac{\partial}{\partial x} [x^2 (1 + \xi) v] = 0, \quad (7)$$

where  $v(a, x)$  denotes the mean relative velocity of pairs at separation  $x$  and epoch  $a$ . Equation (7) can be rewritten in terms of the mean 2-point function,  $\bar{\xi}$ , by defining a new dimensionless pair velocity,  $h(a, x) \equiv -v/\dot{a}x$ ; this yields (Nityananda & Padmanabhan 1994)

$$\frac{\partial D}{\partial A} - h(A, X) \frac{\partial D}{\partial X} = 3h(A, X) \quad (8)$$

where we have introduced the variables  $D = \ln(1 + \bar{\xi})$ ,  $A = \ln a$  and  $X = \ln x$ .

We will now assume  $h \equiv h[\bar{\xi}(a, x)]$ ; *i.e.*  $h$  depends on  $a$  and  $x$  only through  $\bar{\xi}$ . This is a standard assumption in the current literature (Hamilton et al. 1991; Nityananda & Padmanabhan 1994; Mo et al. 1995; Padmanabhan 1996a; Padmanabhan & Engineer 1998) and appears reasonably validated by numerical simulations. Our analysis and results depend on this assumption, which may, of course, be looked upon as another way of truncating the BBGKY hierarchy. In the present work, this assumption will be treated as a basic postulate, validated by simulations; we will not address the question of its limits of validity.

Given the above assumption,  $h \equiv h[\bar{\xi}(a, x)]$ , one can integrate equation (8) to obtain its general solution (Nityananda & Padmanabhan 1994), subject to the condition that this reduce to the form  $\bar{\xi} \propto a^2$  for  $\bar{\xi} \ll 1$ . The characteristics of equation (8) satisfy the constraint

$$x^3 [1 + \bar{\xi}(a, x)] = l^3, \quad (9)$$

where  $l$  is some other length scale. In the linear regime,  $\bar{\xi} \ll 1$  and  $l \approx x$ . At later stages of the evolution, however, as  $\bar{\xi}$  increases, equation (9) shows that the scale  $x$  becomes smaller and smaller as compared to the scale  $l$ . Thus, the evolution of clustering at a scale  $x$  is determined by the original linear power spectrum at the (larger) scale  $l$ ; this suggests that the true non-linear correlation function  $\bar{\xi}(a, x)$  at some scale  $x$  can be expressed in terms of the linear correlation function  $\bar{\xi}_L(a, l)$ , evaluated at a different scale  $l$ . The general solution is expressible as a non-linear scaling relation between  $\bar{\xi}(a, x)$  and  $\bar{\xi}_L(a, x)$ , with  $l$  and  $x$  related by equation (9); this can be written as (Nityananda & Padmanabhan 1994)

$$\bar{\xi}_L(a, l) = \exp \left[ \frac{2}{3} \int^{\bar{\xi}(a, x)} \frac{dz}{h(z)(1+z)} \right], \quad (10)$$

with  $l = x[1 + \bar{\xi}(a, x)]^{1/3}$ .

Given the form of  $h(\bar{\xi})$ , one can now relate the non-linear correlation function to the linear one; this form is usually obtained from simulations. However, it can also be shown from general theoretical arguments (Padmanabhan 1996a) that  $\bar{\xi}(a, x)$  can be expressed in terms of  $\bar{\xi}_L(a, l)$  by the relations

$$\begin{aligned} \bar{\xi}_L(a, l) & \quad (\bar{\xi}_L, \bar{\xi} \ll 1) \\ \bar{\xi}(a, x) & \propto \bar{\xi}_L(a, l)^3 \quad (1 < \bar{\xi} < 200, 1 < \bar{\xi}_L < 5.85) \\ \bar{\xi}_L(a, l)^{3/2} & \quad (200 < \bar{\xi}, 5.85 < \bar{\xi}_L), \end{aligned} \quad (11)$$

if we confine ourselves to models with  $\bar{\xi}_L > 0$  everywhere. More exact fitting functions, valid over the entire range of  $\bar{\xi} > 0$ , can be obtained from simulations (Hamilton et al. 1991; Peacock & Dodds 1994; Mo et al. 1995), for example, the functional form given by Hamilton et al. (1991):

$$\bar{\xi}(a, x) = \frac{z + 0.358z^3 + 0.0236z^6}{1 + 0.0134z^3 + 0.0020z^{9/2}}, \quad (12)$$

where  $z = \bar{\xi}_L(a, l) \equiv a^2 \bar{\xi}_{\text{in}}$ , and  $\bar{\xi}_{\text{in}}$  is the initial mean correlation function.

Thus, given an initial power spectrum  $P_{\text{in}}(k)$ , one can determine the average correlation function (and, hence, the power spectrum, using equation (5)) at any epoch, by simply using equations (4), (9) and the non-linear scaling relation (12).

There is, however, one serious problem with these fitting functions. The original analysis, leading to equations (8) and (9) made no assumptions regarding the sign of  $\bar{\xi}_L$  and is valid even if  $\bar{\xi}_L < 0$  at some scales. The fitting functions, however, *require* that the linear mean correlation function be everywhere positive. This is partly due to the fact that standard power spectra used in cosmology (such as the CDM power spectrum and its variants) do have positive definite correlation functions and there appeared to be no compulsion to consider the case of negative  $\bar{\xi}$ . We feel, however, that this situation is unsatisfactory for two reasons :

(i) The NSR has proved to be a valuable tool in understanding the physics of gravitational clustering and — in particular — the transfer of power between two scales. The basic question in the study of transfer of power is how a sharply peaked power spectrum evolves due to gravitational clustering. To answer this question using the NSR, we need to generalise it for negative values of the mean correlation function.

(ii) As we will show in section 4, observations cannot rule out the existence of negative  $\bar{\xi}$  over certain ranges of scales in the universe. Since this feature has very important implications, it should be taken seriously, without any theoretical prejudice.

It should be noted that attempts have also been made to write NSRs directly for the function  $\Delta^2(k) \equiv k^3 P(k)/(2\pi^2)$  in frequency space (Peacock & Dodds 1994; Peacock & Dodds 1996), rather than in terms of the mean correlation function  $\bar{\xi}$ , as discussed above. In such Fourier space NSRs, one writes

$$\Delta_{NL}^2(k_{NL}) = F[\Delta_L^2(k_L)] , \quad (13)$$

with the wavenumbers  $k_{NL}$  and  $k_L$  related by

$$k_L = [1 + \Delta_{NL}^2(k_{NL})]^{-1/3} k_{NL} . \quad (14)$$

For example, Peacock & Dodds (1994) give the following form for the function  $F$  (we have put  $g(\Omega) = 1$  in their expression, for the case of an  $\Omega = 1$ , matter-dominated Universe) :

$$F(x) = x \left[ \frac{1 + 0.2\beta x + (Ax)^{\alpha\beta}}{1 + ([Ax]^\alpha/[11.68x^{1/2}])^\beta} \right]^{1/\beta} , \quad (15)$$

with  $A = 0.84$  and  $\alpha = \beta = 2$ . Since  $\Delta^2(k) \geq 0$ ,  $k_{NL} \geq k_L$ , and power is transferred from small to large wavenumbers, i. e. from large to small scales. However, no mathematical basis presently exists for scaling relations for quantities other than the mean correlation function  $\bar{\xi}$ ; this is discussed in more detail in the next section.

### 3 NSRS FOR DIFFERENT STATISTICAL INDICATORS.

We discuss, in this section, a few important general issues related to the role of  $\bar{\xi}$  in the context of non-linear scaling relations (NSRs). Throughout this paper we will use the term NSR to mean a relationship between the exact value of some statistical indicator  $Q$  (which could be  $\bar{\xi}, \xi, \Delta \equiv k^3 P(k)/2\pi^2$ , etc) to its value  $Q_{lin}$  as given by linear theory, in the form  $Q(x) = F[Q_{lin}(l)]$  with  $l = (1 + Q)^{1/3} x$ , where  $x$  and  $l$  are two scales and  $F$  is a prescribed function. In the case of Fourier space quantities like  $\Delta(k)$  or  $P(k)$ , we will, of course, interpret  $x$  and  $l$  in term of  $k^{-1}$  for the two scales.

There are two broad perspectives one can take regarding such NSRs. In the first approach, NSRs can be treated as a set of relations which are of considerable practical utility

in studying the evolution of clustering. In this approach, they are merely convenient sets of approximate rules by which non-linear quantities can be obtained from linear ones, thereby facilitating a comparison between theory and observations. Taken in this spirit, the key issues in this area are only the accuracy of the fitting functions  $F$  for the NSR, the dependences on various parameters and, finally, given an NSR for a particular quantity  $Q$ , how best to write down (purely as fitting functions) NSRs for other statistical indicators. There are no fundamental issues; NSRs exist entirely by accident, but are of use in studying non-linear structure formation.

The above attitude, however, is tantamount to sweeping the entire issue under a carpet, without much investigation. In fact, it appears that NSRs are indicative of a key (and not completely understood) feature of non-linear gravitational clustering and hence need to be investigated thoroughly. In support of this view, we would like to stress the following :

(i) Though the original NSR for  $\bar{\xi}$  was obtained as a fitting function to simulation data by Hamilton et al. (1991), the work by Nityananda and Padmanabhan (1995, hereafter NP) clearly spelled out its theoretical origins. In particular, this work showed that (a) there exists an exact equation satisfied by  $\bar{\xi}$  whose integral curves lead to the relation  $l^3 = x^3(1 + \bar{\xi})$  (b) the only key assumption which is needed to obtain the NSR for  $\bar{\xi}$  is that  $h$  is a function of  $\bar{\xi}$  alone.

(ii) So far, there has been no evidence for any mathematical foundation for NSRs for other statistical indicators like  $\Delta^2$  or even  $\xi$ . In fact, there is no analogue for local differential equations like equation (25) in NP, for other statistical indicators like  $\Delta^2$  or the power spectrum. Thus, NSRs appear to exist *only* for the mean correlation function  $\bar{\xi}$ , implying that  $\bar{\xi}$  enjoys a special status as an indicator of clustering. The NSRs given in the literature for  $\Delta(k)$ , for example, arose purely as an afterthought, guided by the nature of the NSR for  $\bar{\xi}$ .

(iii) Given the form of  $\bar{\xi}$ , it is possible to obtain  $\Delta^2$  and  $\xi$  by simple analytic procedures. However, *if* the NSR is true for  $\bar{\xi}$ , it can be shown that such a similar NSR — as defined above — *cannot* be exactly satisfied for  $\Delta^2$  or  $\xi$ . While there have been attempts in the literature to obtain fitting functions for  $\Delta_{NL}^2$  in terms of  $\Delta_L^2$  (Peacock & Dodds 1994, 1996), along the lines of the NSR for  $\bar{\xi}$ , such exercises do not have the same level of fundamental validity as the NSR for the mean correlation function.

The above distinction is not usually of great relevance for the following reasons : First, if the power spectrum is a smooth, mildly varying, power law with adequate asymptotic properties, an NSR for  $\bar{\xi}$  will lead to similar approximate relations for  $\Delta^2$ ,  $\xi$ , etc. Which of them is used in a specific context — assuming that this is all one uses the NSR for — is purely a matter of convenience. Further, since any fitting function is only approximate, the fact that some NSRs are more approximate than others is often not relevant in practical situations.

The situation, however, is very different when one moves

away from simple-minded power spectra and considers a more generic situation. The original analysis of NP (which provided the theoretical foundation for NSRs) did not make any assumptions about the nature of the initial mean correlation function or the power spectrum. Consider, for example, the case of a power spectrum which is sharply peaked around some value, such as a Dirac delta function or a Gaussian. For such a power spectrum,  $\Delta^2$ ,  $\xi$  and  $\bar{\xi}$  have very different shapes. In particular,  $\bar{\xi}$  is quite flat at small scales and decreases towards large scales, while  $\Delta^2$  is close to zero at small scales. Since the shapes of  $\bar{\xi}$  and  $\Delta^2$  are different, the evolution predicted by using the NSR in real space and Fourier space will be totally different. This can be seen very clearly in the case of a power spectrum which is a Dirac delta function or for any power spectrum with a cutoff at short wavelengths (i.e.  $P(k) = 0$  for  $k > k_c$ , with some finite  $k_c$ ). It is precisely in such situations that the mathematical foundations for the two NSRs come into question and, given the fact that the NSR for  $\bar{\xi}$  has a theoretical basis while that for  $\Delta^2$  is only a fitting function, it is the former which must be used when the results from the two differ. Numerical simulations of peaked power spectra (Bagla & Padmanabhan 1997) have shown that, even in such cases, power is indeed transferred from larger to smaller scales, as indicated by the NSR in real space. In fact, such power transfer to scales smaller than the cutoff in the power spectrum *would be impossible* if the Fourier space NSR were correct, simply from the fact that the power spectrum at a given value of  $k$  is a continuous function of the scale factor (this result is true for any power spectrum with a short wavelength cutoff, not merely a peaked power spectrum). While the power spectra of Bagla & Padmanabhan (1997) were exponentially damped beyond a small range of scales, and hence did not have a sharp cutoff, the fact that these simulations did show the transfer of power to small scales, as expected from the  $x$ -space NSR, also indicates that the correct NSR to use is the one for real space, *not* Fourier space.

We note finally, that even if theoretical foundations indeed existed for both forms of the NSR and the two yet yielded different results for the same initial power spectrum / correlation function, one should take resort to dynamical tests such as N-body simulations to ascertain which of the two is applicable. In fact, such a situation might well prove interesting as it would provide information about the dynamical correctness of the two scaling relations.

The strength of the present work is that its conclusions do not depend on the form of the NSR but arise entirely from its existence. The synthetic NSR of the last section of the paper (and the appendix) is used solely to illustrate the results. It should be emphasised that we will show analytically that the results are independent of the form of the NSR. In the spirit of the above discussion, we will work in  $x$ -space and not in Fourier space during the analysis.

#### 4 AN ASIDE : OBSERVATIONAL CONSTRAINTS

As mentioned earlier, negative values of  $\bar{\xi}$  are *not* mathematically forbidden and, in fact, arise naturally for sufficiently peaked forms of the power spectrum. We, will, in this section, briefly review observational constraints to investigate whether present-day observations of the correlation function or the power spectrum rule out the existence of negative  $\bar{\xi}$  in our universe.

At very large scales,  $l1000h^{-1}$  Mpc, the CMBR anisotropy measurements provide a constraint on the slope and amplitude of the power spectrum;  $\bar{\xi}_L$  is probably positive in this range. At small scales,  $1h^{-1}$  Mpc  $l200h^{-1}$  Mpc, the galaxy-galaxy correlation function provides the shape and amplitude of the baryonic component of the universe. This can be converted into providing some handle on the underlying dark matter distribution and the results are consistent with a positive mean correlation function. At still smaller scales, information on the power spectrum arises from the study of abundances of bound structures (like quasars, damped Lyman- $\alpha$  systems, etc) and it is very likely that the mean correlation function is again positive at these scales. There is, however, a range of scales ( $\sim 200h^{-1}$  Mpc –  $1000h^{-1}$  Mpc) which are not directly probed by present observations and one can easily construct power spectra for which the correlation function goes negative in this range. Of course, while doing this, one has to also ensure that the additional power does not create any observable consequences at other scales.

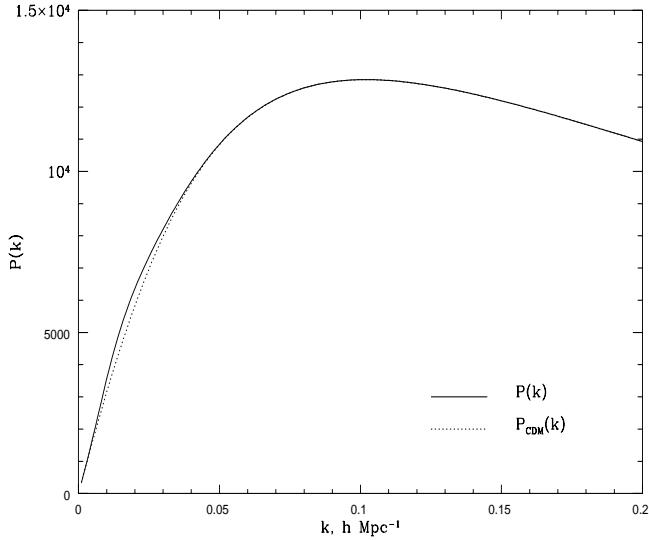
We demonstrate, next, (by an explicit example) that it is possible to construct power spectra which satisfy the above observational constraints and yet yield negative  $\bar{\xi}$ . As mentioned above, the observations allow negative values of  $\bar{\xi}$  over the range  $200h^{-1}$  Mpc  $< x < 1000h^{-1}$  Mpc. Consider the COBE-normalised CDM power spectrum of Bardeen et al. (1986) (eq. G3)

$$P_{\text{CDM}}(k) = Ak \left[ \frac{\ln[1 + 2.34q]}{2.34q} \right]^2 \times [1 + 3.89q + (16.1q)^2 + (5.46q)^3 + (6.71q)^4]^{-1/2}, \quad (16)$$

with  $q = k/(\Omega h^2 \text{Mpc}^{-1})$  (using  $\theta = 1$  in eq. (G3), corresponding to photons and three types of relativistic neutrinos). The value of  $A$  is set by COBE normalisation to be about  $(24h^{-1} \text{Mpc})^4$ . A good fit to the average correlation function  $\bar{\xi}_{\text{CDM}}$  for the above spectrum is given by Hamilton et al. (1991)

$$\bar{\xi}_{\text{CDM}}(x) = 0.51 \left[ \ln \left( 1 + \frac{5}{x} \right) \right]^3 \frac{\ln(1 + x/12)}{x/12} \times \frac{1 + 0.394x + 0.00316x^2}{1 + 0.142x + 0.00129x^2}, \quad (17)$$

where  $x$  is in  $\Omega^{-1}h^{-2}$  Mpc and  $h$  is the Hubble parameter.



**Figure 1.** The figure shows the CDM power spectrum (dotted line) and the net power spectrum  $P(k) = P_{\text{CDM}}(k) + P_2(k)$  (solid line). It can be seen that the two are different only over a small range in  $k$ -space.

We will work in an  $\Omega = 1$  Universe and add to the CDM spectrum, a peaked spectrum of the form

$$P_2(k) = A' k^4 e^{-\lambda k} ; \quad (18)$$

the correlation function corresponding to  $P_2(k)$  is

$$\bar{\xi}_2(x) = -\frac{72A'\lambda}{\pi^2} \frac{3x^2 - 5\lambda^2}{(x^2 + \lambda^2)^5} , \quad (19)$$

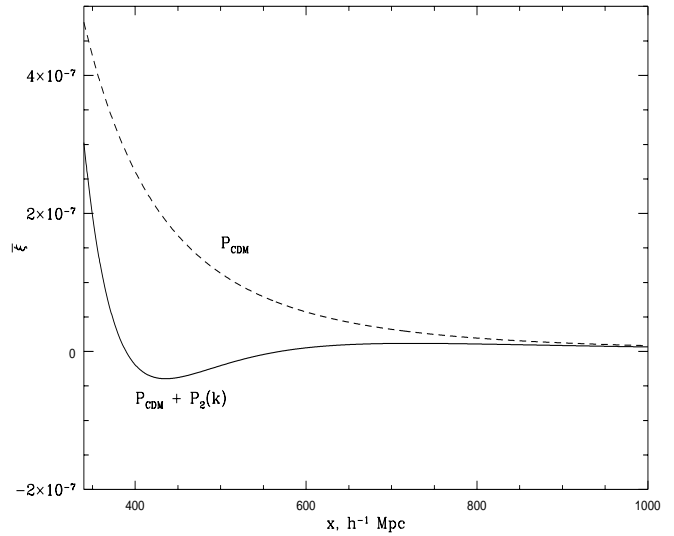
which clearly goes negative for  $x > (5/3)^{1/2} \lambda$ .

The net power spectrum  $P(k) = P_{\text{CDM}}(k) + P_2(k)$  (with  $A' \sim (841 h^{-1} \text{ Mpc})^4$  and  $\lambda = 250 h \text{ Mpc}^{-1}$ ) is plotted in figure (1) (solid line), along with the CDM spectrum (dotted line). It can be seen that the deviation of the net spectrum from the CDM spectrum occurs only over an extremely small range of  $k$  values and this deviation is itself extremely small.

Figure (2) shows the average correlation function  $\bar{\xi}(x)$ , corresponding to  $P(k)$ . It can be seen that  $\bar{\xi}(x)$  becomes negative at  $x \sim 390 h^{-1} \text{ Mpc}$  and then returns to positive values at  $x \sim 600 h^{-1} \text{ Mpc}$ . In the region probed by COBE ( $1000 h^{-1} \text{ Mpc} - 3000 h^{-1} \text{ Mpc}$ ),  $\bar{\xi}$  is indistinguishable from the CDM correlation function; this is due to the extremely rapid fall-off in  $\bar{\xi}_2(x)$ .

Thus, even broad band power spectra can yield negative correlation functions. The example shown in figure (1) is just one of many possible ways to construct such power spectra, which agree with the current observations. It is hence clearly important to understand the effects of such anti-correlated regions on the evolution.

We emphasise that the power spectra we have constructed



**Figure 2.** The average correlation function  $\bar{\xi}(x)$ , for the two power spectra of figure (1). The correlation function corresponding to the total power spectrum,  $P(k)$ , can be seen to go negative at  $x \sim 390 h^{-1} \text{ Mpc}$ . The two correlation functions approach each other beyond  $x \sim 800 h^{-1} \text{ Mpc}$ .

will satisfy all observational constraints currently available but will introduce some new features in the evolution of the universe. The purpose of doing this is only to stress how little we know about our universe; we have hence not worried about whether there exist suitable physical processes which will lead to such a power spectrum.

## 5 CONTINUITY ARGUMENTS AND BOUND ON $\bar{\xi}$

We will now consider the evolution of  $\bar{\xi}$  with time, at a given scale  $x$ , and show that the mere existence of a negative value of  $\bar{\xi}_L$ , at some scale  $l$ , *implies* an upper limit to the value of  $\bar{\xi}$  which can be reached at any scale smaller than  $l$ ; further, the maximum value  $\bar{\xi}_{\text{max}}$  at any scale is a function of the scale itself and the location  $L_0$ , where  $\bar{\xi}_L$  first crosses zero. This effect also results in the correlation function acquiring a  $x^{-3}$  dependence at late stages of the evolution.

We begin by noting that the arguments leading to the paradigm for transfer of power (see Section 2) made no assumption about the sign of  $\bar{\xi}$ ; these arguments, as well as equations (9) and (10), thus hold good for the negative regime as well. Next, the relation

$$l = x [1 + \bar{\xi}(a, x)]^{1/3} , \quad (20)$$

which governs the transfer of power from one scale to another, implies that a given value of  $\bar{\xi}$  can arise at some scale  $x$  *only* by the transfer of power from a single scale  $l$ . For  $\bar{\xi} > 0$ ,

$x < l$ , *i.e.* power flows to smaller scales, while, if  $\bar{\xi} < 0$ ,  $x > l$ , *i.e.* power transfers to larger scales. We stress that equation (20) made no assumptions about the sign of  $\bar{\xi}$  except, of course, the obvious condition,  $(1 + \bar{\xi}) > 0$ .

Let  $L_0$  be the first scale at which  $\bar{\xi}_L$  crosses zero and let  $x_0$  be the scale of interest, with  $x_0 < L_0$ . Initially, in the linear regime,  $\bar{\xi} \ll 1$ ,  $x_0$  is influenced by scales  $l$  such that  $l \approx x_0$ . At later epochs, as  $\bar{\xi}$  increases,  $x_0$  is influenced by larger and larger scales, upto the scale  $L_0$ . Power from the scale (infinitesimally larger than)  $L_0$ , however, must be transferred to scales *larger* than  $L_0$ , since  $\bar{\xi}_L < 0$  here; clearly, this scale ( $l = L_0 + \Delta l$ ) *cannot* influence  $x_0$ . However, as has already been emphasised, a given value of  $\bar{\xi}$  can arise at the scale  $x_0$  only by the transfer of power from a single scale; this immediately implies that  $\bar{\xi}(x_0)$  can *never* exceed the value  $\bar{\xi}_o$ , given by

$$\bar{\xi}_o = \left[ \frac{L_0}{x_0} \right]^3 - 1, \quad (21)$$

since this would require power transfer from the scale  $L_0 + \Delta l$  to  $x_0$ , which is impossible. Similarly, if  $\bar{\xi}_L < 0$  over some range of  $l$ , say, from  $l_1$  to  $l_2$ , then no value of  $\bar{\xi}$  which satisfies the relation

$$\bar{\xi} = \left[ \frac{l'}{x_0} \right]^3 - 1 \quad (22)$$

can be reached at  $x_0$ , for  $l'$  between  $l_1$  and  $l_2$ .

The correlation function, however, is a continuous function; clearly, then, if some values of  $\bar{\xi}$  cannot be reached,  $\bar{\xi}$  *must* be bounded from above by the lowest of these values. Thus, the maximum value which  $\bar{\xi}$  can take at some scale  $x$  ( $x < L_0$ ) is given by

$$\bar{\xi} = \left[ \frac{L_0}{x} \right]^3 - 1 \quad (23)$$

where  $L_0$  is the smallest scale at which  $\bar{\xi}_L$  crosses zero. Thus, at late times,  $\bar{\xi} \gg 1$ , we must have

$$\bar{\xi} \propto x^{-3} \quad (24)$$

for  $x$  smaller than  $L_0$ . This is a generic result, independent of the exact form of the initial correlation function, except for the requirement that it become negative at some scale.

Note that  $\bar{\xi}(x) \propto x^{-3}$  over some range of scales implies that  $\xi(x) = 0$  over this range (and vice-versa). For example, if  $\xi(x) \approx 0$  for  $x > L$ , then

$$\bar{\xi}(x > L) = \frac{3}{x^3} \left[ \int_0^L y^2 \xi(y) dy + \int_L^x y^2 \xi(y) dy \right] \quad (25)$$

$$\simeq \frac{3}{x^3} A, \quad (26)$$

where  $A = \int_0^L y^2 \xi(y) dy$  is a constant. Thus,  $\bar{\xi}(x) \propto x^{-3}$ , for  $x > L$ , if  $\xi(x) \approx 0$  here. The result that the relation  $\bar{\xi}(x) \propto x^{-3}$  spreads to smaller scales at late times can be then seen to imply that  $\xi(x)$  becomes negligible at smaller and smaller scales at later and later epochs.

It may appear strange that the mere existence of negative  $\bar{\xi}_L$  has far-reaching effects on the evolution. However, it must be emphasised that the positive and negative  $\bar{\xi}$  regimes are *not* equivalent, due to the lower bound of  $(-1)$  on  $\bar{\xi}$  in the negative regime; no such bound exists for  $\bar{\xi} > 0$ . This implies an asymmetry in the very structure of the equations, if one uses  $\bar{\xi}$  as a variable. In fact, this also indicates that  $\text{Log}[1 + \bar{\xi}]$  is probably the correct variable which should be used; in this case, the above asymmetry does not arise, since  $\text{Log}[1 + \bar{\xi}]$  takes all values from  $-\infty$  to  $+\infty$ .

In general, linear theory ceases to be valid when  $|\bar{\xi}|$  is of the order of unity. However, while using the NSR, one maps the linearly extrapolated value  $\bar{\xi}_L$  to the actual value of  $\bar{\xi}$ . This poses no mathematical problems for  $\bar{\xi} > 0$ , since arbitrarily large positive values of linearly extrapolated  $\bar{\xi}_L$  are *mathematically* allowed and map to still higher values of actual  $\bar{\xi}$  through the NSR; the situation is quite different in the negative regime since the actual  $\bar{\xi}$  is constrained to be greater than  $(-1)$ . This asymmetry appears to play a role in situations in which negative correlations exist.

We mentioned in section 3 that power transfer to scales smaller than the cutoff in the initial power spectrum is impossible for power spectra with sharp cutoffs, if the Fourier space NSR is correct. This can be seen by an argument similar to the one discussed above. Consider an initial power spectrum with a short wavelength cutoff at some scale  $k_o$ . In Fourier space NSRs (see equations (13) to (15)), the power  $P(k)$  at a wavenumber  $k$  at some epoch originates at a fixed scale  $k_l$ , with  $k_l$  and  $k$  related by (Peacock & Dodds 1994)

$$k^3 = k_l^3 [1 + \Delta^2(k)], \quad (27)$$

with  $\Delta^2(k) = k^3 P(k)/(2\pi^2)$ . Since  $\Delta^2(k) \geq 0$ , clearly  $k_l \leq k$ . If we now consider a wavenumber  $k$  beyond the cutoff in the initial spectrum, with  $k > k_o$ , there exists a range of wavenumbers between  $k$  and  $k_o$ , where  $P_{in}(k) = 0$  and the power at  $k$  is *then forbidden to take values which could originate between  $k_o$  and  $k$* . Since the power spectrum is a continuous function of the scale factor, if  $P(k)$  is not allowed to take a set of values, it also cannot take values larger than the lowest value of this set. Thus, even in Fourier space, it can be seen that an upper bound exists on the power spectrum at large wavenumbers (*i.e.* small scales)  $k > k_o$ , for initial power spectra which cut off at some wavenumber  $k_o$ . We note that such power spectra do, in fact, give rise to negative correlation functions; one could thus have instead used the real space NSR to demonstrate the upper bound on  $\bar{\xi}$ . However, there are also other power spectra such as Gaussians, which do produce negative  $\bar{\xi}$  even without sharp cutoffs in the initial spectrum; in such cases, the upper bounds are not as obvious from a Fourier space analysis but can be clearly seen in real space. Of course, as discussed in section 3, different results may then be obtained for the late-time behaviour of  $P(k)$  and  $\bar{\xi}$  from the two NSRs, especially in the case of peaked initial power spectra. In such situations,  $\bar{\xi}$  is picked out as the statistical indicator to be used in the

NSR, by the analysis of NP which provides a theoretical basis for the real space scaling relations.

## 6 NSR FOR NEGATIVE $\bar{\xi}$

We require an NSR for the negative  $\bar{\xi}$  regime to explicitly illustrate the results of the preceding section. However, the form of the NSR given in equation (12) tacitly assumes that  $\bar{\xi}_L(a, l) > 0$  for all  $l$ ; as mentioned earlier, this requirement has no fundamental significance. The above form, however, can be derived from theoretical arguments related to spherical collapse, in the regime  $\bar{\xi} > 0$  (Padmanabhan 1996a). This suggests that the NSR for the case  $\bar{\xi} < 0$  may also be obtained by an analysis of a spherically symmetric model, with negative correlations at some scale. This will allow us to guess at an ansatz for  $h(\bar{\xi})$ , based on the relation between  $h$  and the density contrast  $\delta$  for this model. This form for  $h(\bar{\xi})$  will then be used in equation (10) to obtain an NSR in the negative  $\bar{\xi}$  regime.

We will use the ansatz,

$$h(\bar{\xi}) = \frac{2}{3}\bar{\xi} + \frac{1}{6}\bar{\xi}^2 \quad (-1 < \bar{\xi} < 0), \quad (28)$$

for  $h(\bar{\xi})$  (see Appendix for motivation). Substituting for  $h(\bar{\xi})$  in equation (10), we obtain, after some algebra

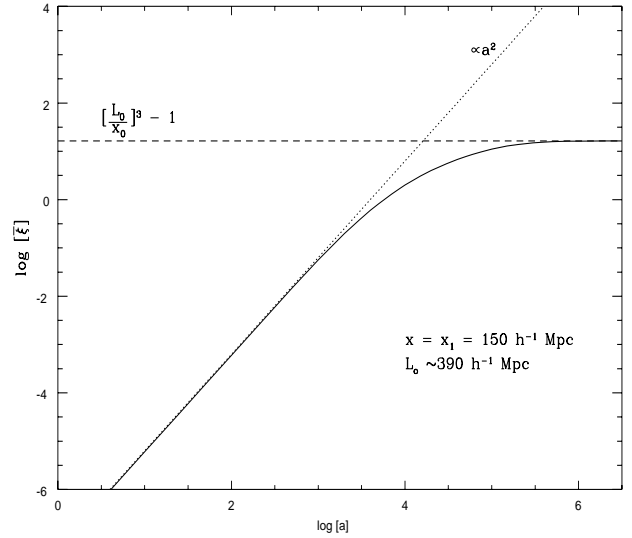
$$\bar{\xi}_L(a, l) = \frac{\bar{\xi} (1 + \bar{\xi}/4)^{1/3}}{(1 + \bar{\xi})^{4/3}}. \quad (29)$$

The inverse mapping, giving  $\bar{\xi}$  in terms of  $\bar{\xi}_L$ , can be fitted to better than 5% by the relation

$$\bar{\xi} = \frac{\bar{\xi}_L + 0.88\bar{\xi}_L^2 + 2.086\bar{\xi}_L^3}{1 + 1.798\bar{\xi}_L^2 - 2.086\bar{\xi}_L^3} \quad (-1 < \bar{\xi} < 0) \quad (30)$$

Equations (9) and (30) implicitly determine  $\bar{\xi}(a, x)$  in terms of the linearly evolved correlation function,  $\bar{\xi}_L(a, l)$  for all values, positive and negative, of  $\bar{\xi}_L$ . We mention that the above NSR, equation (30), will be used solely for the purpose of illustrating the results of Section (4), since these results are independent of the exact form of the NSR. The true mapping between  $\bar{\xi}$  and  $\bar{\xi}_L$  in this regime should, of course, be determined from N-body simulations.

We will use the net power spectrum  $P(k)$  of figure (1) (which is allowed observationally) to demonstrate the effects of negative  $\bar{\xi}$ . The results are shown in figures (3) and (4), in which  $\bar{\xi}$  is plotted against the scale factor, for two scales,  $x_1 = 150h^{-1}$  Mpc and  $x_2 = 20h^{-1}$  Mpc. It can be seen that, in both cases,  $\bar{\xi}$  does not continue to increase with  $a$  (or, equivalently, with time), but instead flattens out at late times and approaches a value of  $(L_0/x_0)^3 - 1$  (dotted lines), as predicted by the analysis. Here,  $L_0$  is the scale at which  $\bar{\xi}_L$  first goes negative and  $x_0$  is the scale under consideration. We emphasise that the mapping for negative  $\bar{\xi}$  has no influence on the results; any mapping for this regime would have yielded the same results since these stem entirely from the *existence* of negative values of the correlation function.



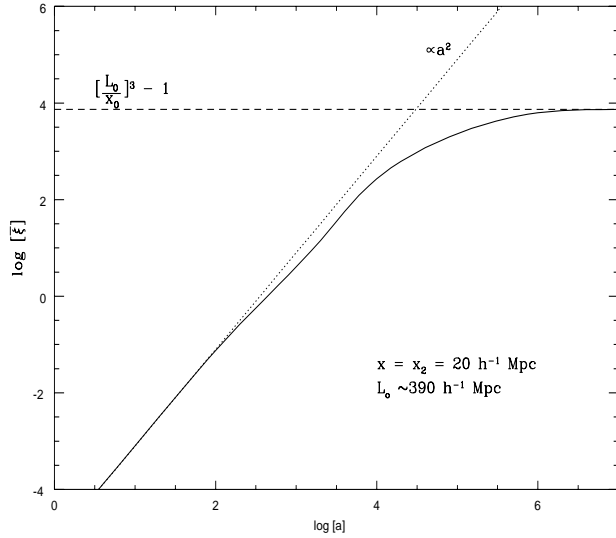
**Figure 3.** The figure shows  $\bar{\xi}$  as a function of scale factor, at  $x_1 = 150h^{-1}$  Mpc, for the power spectrum  $P(k)$  discussed in the text. The dotted line corresponds to  $\bar{\xi} \propto a^{-2}$ .

Figure (5) shows a log-log plot of  $\bar{\xi}$  versus  $x$  at four different epochs; the solid line has a slope of  $-3$ . It can be seen that more and more scales acquire the  $\bar{\xi} \propto x^{-3}$  behaviour with time. Scales with  $x3h^{-1}$  Mpc (*i.e.*  $\bar{\xi}2 \times 10^6$ ) have not as yet (upto  $a = 5 \times 10^5$ ) been influenced by the region with  $\bar{\xi}_L < 0$  and will hence only acquire the  $x^{-3}$  slope at later epochs. It should be emphasised that the above discussion regarding the upper bound on  $\bar{\xi}(x)$  is applicable *only* for scales  $x$  which are *smaller* than the scale  $L_0$  at which the initial mean correlation function  $\bar{\xi}_L$  first goes negative and not for scales  $x > L_0$ . We do not discuss the evolution of  $\bar{\xi}(x)$  for  $x > L_0$  here, since this would depend on the exact form of the scaling relation in the negative  $\bar{\xi}$  regime and we presently only possess a rather synthetic NSR in this region.

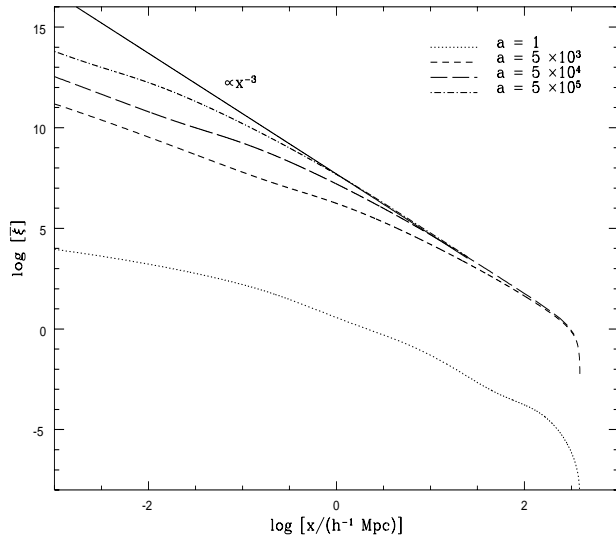
Finally, we note that the “asymptotic” behaviour of  $\bar{\xi}$  does not arise at any particular value of  $\bar{\xi}$  but depends only on the scales in question,  $x_0$  and  $L_0$ . For example, in figure (3), the asymptotic behaviour arises when  $\bar{\xi} \sim 16$  (*i.e.* in the quasi-linear regime) while, in figure (4), this behaviour arises when  $\bar{\xi} \sim 7 \times 10^3$  (*i.e.* in the non-linear regime).

The mean correlation function  $\bar{\xi}$  is known to scale as  $\bar{\xi} \propto a^2$ ,  $a^6$  and  $a^3$  (in 3 dimensions) in the linear, quasi-linear and non-linear regimes of structure formation (Hamilton et al. 1991; Padmanabhan 1996a). However, the existence of negative correlation functions appears to give rise to a *fourth regime* in gravitational clustering, besides the above three regimes. Here,  $\bar{\xi}$  asymptotically approaches a constant value, at any scale shortward of the scale at which  $\bar{\xi}_L$  crosses zero. Clearly, high dynamic range simulations will be needed to test this result; this may well prove difficult, or





**Figure 4.** The figure shows  $\bar{\xi}$  as a function of scale factor, at  $x_2 = 20h^{-1}$  Mpc, for the power spectrum  $P(k)$  discussed in the text. The dotted line corresponds to  $\bar{\xi} \propto a^{-2}$ .



**Figure 5.** The figure shows the mean correlation function  $\bar{\xi}$  plotted against scale  $x$ , for four different epochs (dot-dashed, dotted, short-dashed and long-dashed lines). The solid line has a slope of  $-3$ . It can be seen that more and more scales acquire the  $\bar{\xi} \propto x^{-3}$  behaviour at later and later times.

even prohibitive, in 3-dimensions. Such high dynamic ranges can, however, be attained in 2-D simulations; we note that earlier 2-D simulations (Bagla, Engineer & Padmanabhan 1997) have shown that  $\bar{\xi}$  does not continue to grow as  $a^2$  in the non-linear regime (as expected from stable clustering), but begins to flatten out at late times. It is possible that this is indicative of the existence of a fourth regime, beyond the stable clustering phase. Of course, the present results are quite independent of the stable clustering ansatz, since they stem solely from the existence of the non-linear scaling relation.

We note finally that the evolution of non-linear structure may also be studied using other models; for example, one may view the non-linear correlation function as arising from the superposition of the density profiles of non-linear halos (Seljak 2000; Peacock & Smith 2000). However, NSRs seem to be reasonably validated by numerical simulations; we have hence not considered other models, such as the halo model, here, but have limited ourselves to attempting to understand the scaling relations themselves.

In summary, the mere existence of negative correlation functions have far-reaching effects on the evolution of clustering. They result in an upper bound on  $\bar{\xi}$  at scales smaller than the scale at which  $\bar{\xi}_L$  goes negative and also cause the relation  $\bar{\xi} \propto x^{-3}$  at these scales, at late times. Negative correlation functions are not forbidden observationally; it is hence important to use N-body simulations and/or semi-analytic arguments to obtain a form for the NSR in this regime.

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## 7 APPENDIX

We will use the evolution of a spherically symmetric, compensated void to obtain some insight into the form of  $h(\xi)$ . Consider an initial density perturbation of the form

$$\begin{aligned} -|\delta_1| & \quad (0 < R < R_1) \\ \delta(R) &= \delta_2 \quad (R_1 < R < R_2) \\ 0 & \quad (R_2 < R) , \end{aligned} \quad (31)$$

such that the effects of the under-dense region within  $R_1$  are “compensated” by the over-dense region between  $R_1$  and  $R_2$ . This implies that a shell with radius larger than  $R_2$  does not feel any excess gravitational force, which would cause it to deviate from the smooth background expansion. Such a configuration can be seen to give rise to negative values of the average correlation function.

We initially investigate the asymptotic behaviour of the  $h$  function for such a system, and obtain a fitting function for  $h(\delta)$  when  $\delta < 0$ . These results will be used to argue for a form of  $h(\xi)$ , which can be then used to obtain an NSR between  $\xi$  and  $\xi_L$ , in the negative  $\xi$  regime.

The general equation for the evolution of a spherical shell which encloses a mass  $M$  can be written as

$$\frac{d^2 R}{dt^2} = -\frac{GM}{R^2} , \quad (32)$$

where  $R(t)$  is the proper radius of the shell; the mass  $M$  is a constant, in the absence of shell crossing. The first integral of equation (32) is

$$\frac{1}{2} \left( \frac{dR}{dt} \right)^2 = \frac{GM}{R} + E . \quad (33)$$

The constant  $E$  gives the total energy of the shell; clearly,  $E > 0$  for a shell with radius,  $R < R_2$  while  $E = 0$  for a shell with radius,  $R > R_2$ . We will set up the initial conditions

by assuming that the shell has radius  $R_i$  at the initial time,  $t_i$  and moves with the general background expansion at this time, with zero peculiar velocity. The total energy of the shell is then given by (Padmanabhan 1996b)

$$E = -K_i \delta_i , \quad (34)$$

where  $K_i = (1/2)H_i^2 R_i^2$  is the initial kinetic energy,  $\delta_i$  is the initial density contrast and  $H_i$  is the Hubble parameter at the initial time; note that  $\delta_i < 0$ , so that  $E > 0$ .

The solution to equation (32), with these initial conditions, can be written in the parametric form

$$R = \frac{GM}{2E} [\cosh \theta - 1] \quad (35)$$

$$t = \frac{GM}{(2E)^{3/2}} [\sinh \theta - \theta] . \quad (36)$$

The peculiar velocity,  $v$ , defined as  $v = a\dot{x}$ , is

$$v = \frac{d}{dt} [ax] - \dot{a}x = \dot{R} - HR . \quad (37)$$

This implies that

$$h \equiv -\frac{v}{HR} = 1 - \frac{\dot{R}}{RH} . \quad (38)$$

For an  $\Omega = 1$ , matter dominated Universe,  $H = (2/3t)$ ; substituting for  $H$  and  $\dot{R}$  in equation (38) yields

$$h = \left[ 1 - \frac{3t}{2R} \sqrt{2 \left( \frac{GM}{R} + E \right)} \right] . \quad (39)$$

We now use equations (35) and (36) to rewrite the above in terms of the parameter  $\theta$  as

$$h = 1 - \frac{3 \cosh(\theta/2)}{2 \sinh(\theta/2)} \left[ \frac{\sinh \theta - \theta}{(\cosh \theta - 1)^{3/2}} \right] . \quad (40)$$

Further, the density contrast  $\delta$  is given by

$$1 + \delta = \frac{9GMt^2}{2R^3} . \quad (41)$$

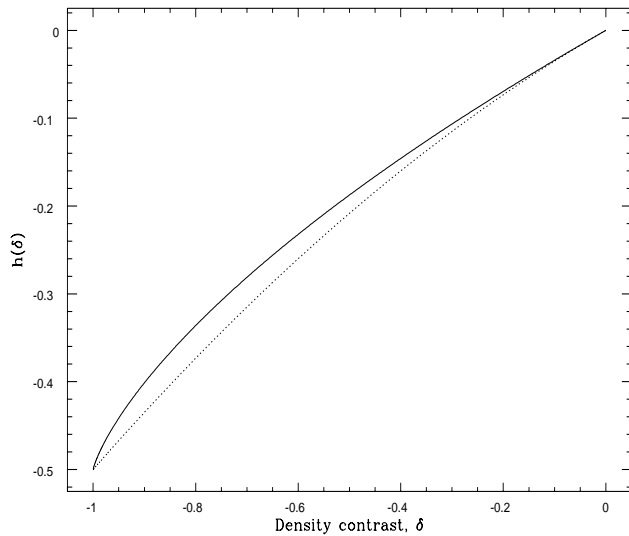
Again replacing for  $R$  and  $t$  in terms of  $\theta$  gives

$$\delta = \frac{9}{2} \frac{(\sinh \theta - \theta)^2}{(\cosh \theta - 1)^3} - 1 . \quad (42)$$

In the limit  $\theta \rightarrow \infty$ ,  $\delta \rightarrow -1$  and  $h \rightarrow -1/2$ . We thus have the interesting result that  $h \rightarrow -1/2$  as  $\delta \rightarrow -1$ .

A good fit to the function  $h(\delta)$ , using equations (40) and (42), is given by  $h(\delta) = \delta/3 - \delta^2/6$  (see figure (6)). The fit satisfies both the linear limit ( $h(\delta) \sim \delta/3$ , for  $|\delta| \ll 1$ ), as well as tends to  $-1/2$  as  $\delta \rightarrow -1$ . The percentage error on the fit is less than 10%.

The preceding analysis of an individual compensated void has yielded a fitting function for  $h(\delta)$ ; we, however, require  $h$  as a function of  $\xi$ , to find the NSR for the negative  $\xi$  regime. There is – in general – no simple relation between the density contrast  $\delta$  of a single lump and the correlation function



**Figure 6.** The figure shows the  $h$  function, plotted versus density contrast,  $\delta$ , for a compensated void (solid line). The dotted line indicates a fit of the form  $h(\delta) = a\delta + b\delta^2$ , with  $a = 1/3$ ,  $b = -1/6$ .

$\bar{\xi}$ . (Purists will even consider the question of relating  $\bar{\xi}$  and  $\delta$  meaningless.) Despite this, relations obtained for single spherical lumps have proved to be generalizable to a description of statistical quantities like  $\bar{\xi}$  (Padmanabhan 1996a; Padmanabhan 1997; Mo & White 1996; Sheth 1998; Engineer, Kanekar & Padmanabhan 2000). In this spirit, and motivated by the relation between  $h$  and  $\delta$  being quadratic, we will attempt an ansatz in which  $h$  is a quadratic function of  $\bar{\xi}$ . In the linear regime, however,  $h = (2/3)\bar{\xi}$ ; this result arises from straightforward application of perturbation theory (Padmanabhan 1996a) and is independent of the sign of  $\bar{\xi}$ . Next,  $\bar{\xi}$  and  $\delta$  are both bounded from below by -1 and  $\bar{\xi} \propto \delta$  in the non-linear regime (Engineer, Kanekar & Padmanabhan 2000). It thus seems reasonable that  $\bar{\xi} \approx -1$  when  $\delta = -1$ ; this immediately implies that  $h \sim -1/2$  when  $\bar{\xi} \approx -1$ . We will hence choose

$$h(\bar{\xi}) = \frac{2}{3}\bar{\xi} + \frac{1}{6}\bar{\xi}^2, \quad (43)$$

similar to the form of  $h(\delta)$  and satisfying the condition  $h \rightarrow -1/2$  as  $\bar{\xi} \rightarrow -1$ . This will be used in equation (10) to obtain an NSR for the negative  $\bar{\xi}$  regime.